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Letter to the Editor

# Post-critical behaviour of Euler and Beck columns resting on an elastic foundation

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## 1. Introduction

Beams on elastic foundations are commonly used in civil engineering problems. They are typically used for modelling beams resting on soils, but are also used in several other applications such as railroad tracks, submerged pipes, etc. The Winkler foundation model represents the simplest form of elastic foundation. In this model, the foundation is treated essentially as an array of closely spaced but non-interacting springs, each having a spring stiffness that equals the foundation modulus divided by the spacing between springs. In most practical applications, this foundation is used to model soil behaviour.

Analyses of beams on foundations have been extensively studied for the linear elastic case. Little attention, however, has been given to their behaviour in the non-linear range. Timoshenko and Gere [1] proposed the solution for simply supported uniform beams resting on Winkler-type foundation. Bowles [2] derived a stiffness matrix for the problem using a conventional beam element supported on discrete springs only at its ends. Ting and Mockry [3] developed a stiffness matrix of an elastic beam on lateral support suitable for the displacement method of analysis. The matrix is determined by deriving the exact solution of the differential equation of the problem. Lai et al. [4] extended the same work for dynamic cases. Eisenberger and Yankelevsky [5,6] used the same approach and developed an exact stiffness matrix for beam on elastic foundation including axial effects. Several researchers have proposed the finite element method using displacement shape functions [7–9]. The finite element formulation, which takes the shear deformation into consideration for analyzing the beams on elastic foundation, has been developed by Aydogan [10] and Dasgupta and Sengupta [11]. Thambiratnam and Zhuge [12] used the same approach to study the vibration and dynamic behaviour of these types of structures. Kaschiev and Mikhajlov [13] used the finite element method as a general numerical technique to solve the problem of elastic beams on tensionless foundations for different loading conditions. Beaufait and Hoadley [14] have accounted the non-linear Winkler foundation in their beam analysis.

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Regarding the instability analysis, Dube and Dumir [15] presented solutions for ascertaining the buckling loads of tapered beams on various types of elastic foundation. Naidu and Rao [16,17] computed the stability parameter of uniform beams resting on a class of two-parameter elastic foundation by solving linear eigenvalue problems. They extended the approach to compute the buckling parameter of beams on a non-linear elastic foundation [18]. Rao and Raju [19] proposed an equivalent Winkler foundation to represent two-parameter elastic foundations. Dutta and Roy [20] summarized various formulation models employed in the field of structural mechanics.

The dynamic stability of structural elements has been well discussed by Bolotin [21]. The stability of a uniform cantilever column subjected to a tip-concentrated subtangential follower force has been analyzed by Kikuchi [22], using a finite element method. Smith and Herrmann [23] have shown that a uniform cantilever column subjected to a follower end force and having dynamic instability is neither stabilized nor de-stabilized by the introduction of a uniform elastic foundation. Shastry and Rao [24] have examined the dynamic stability of Euler columns resting on an elastic foundation through finite element analysis. Rao and Rao [25] have presented the stability analysis results of a cantilever column resting on an elastic foundation under a subtangential follower force at its free end, whereas in Ref. [26], they have presented results for large-amplitude vibrations of a tapered cantilever beam.

The basic equations for the large deformation of cantilever columns under subtangential follower forces considering horizontal foundation reaction are presented in Ref. [25]. When the column is loaded beyond its critical load, it is essential to consider the realistic rotational (normal) foundation reaction instead of the horizontal foundation reaction as being considered in Ref. [25]. The governing equations for small deformation of cantilever columns are found to be the same for both the cases of normal and horizontal foundation reactions. Hence, the critical loads of Ref. [25] are valid for all the subtangential follower forces. The purpose of this article is to present a general formulation for the problem, which takes into account the realistic normal foundation reaction and to provide some observations of interest obtained by using the dynamic criterion for the post-critical behaviour of uniform cantilever columns resting on an elastic foundation under a tip-concentrated subtangential follower force. Though, the formulation of the problem is general (which is valid for all subtangential follower forces), the post-critical loads are presented here only for Euler and Beck columns resting on an elastic foundation.

## 2. Analysis

The formulation of the problem is mainly based on an important relation of the flexure theory (i.e., $M/EI = 1/\rho = \mathrm{d}\theta/\mathrm{d}s$). The quantity $1/\rho$ (the curvature of the deflected axis of the column) characterizes the magnitude of bending deformation, which is proportional to the bending moment, $M$, and inversely proportional to the product $EI$ called the flexural rigidity of the column. The moment–curvature relationship including both the axial and transverse inertia of a uniform cantilever column resting on an elastic foundation subjected to a tip-concentrated subtangential follower force (see Fig. 1) can be written as

$$EI\frac{\mathrm{d}\theta}{\mathrm{d}s} = P\cos(\gamma\alpha)\{Y - Y_a\} - P\sin(\gamma\alpha)\{X - X_a\} + L_{JA} + L_{JT}, \tag{1}$$

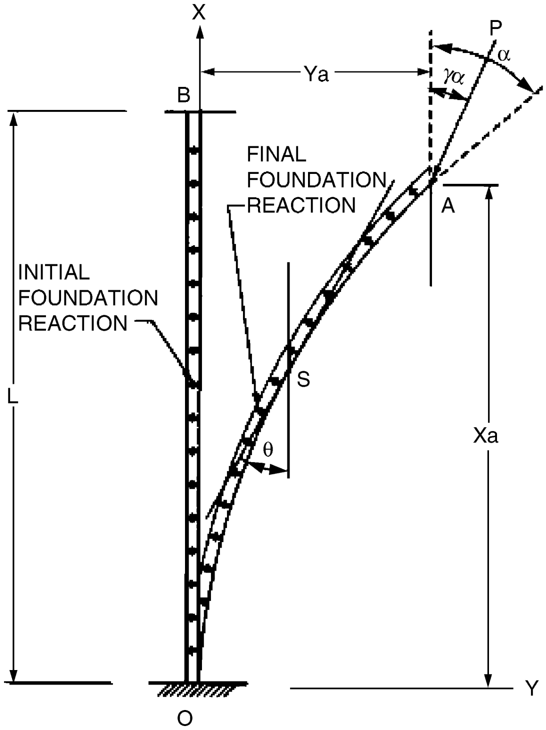


Fig. 1. A cantiliver column on an elastic foundation subjected to a subtangential follower end force. Co-ordinate transformation: $\eta = S/L$; at location A, $\eta = 0$; at location O, $\eta = 1$ ($\gamma = 0$, Euler column; $\gamma = 1$, Beck column).

where

$$L_{JA} = \int_0^s \left\{ku + m\frac{d^2u}{dt^2}\right\}\{Y(s) - Y(\xi)\}\,d\xi, \tag{2}$$

$$L_{JT} = \int_0^s \left\{kv + m\frac{d^2v}{dt^2}\right\}\{X(s) - X(\xi)\}\,d\xi, \tag{3}$$

$$u = X - L + s, \tag{4}$$

$$v = Y, \tag{5}$$

$$X = \int_s^L \cos\theta\,ds, \tag{6}$$

$$Y = \int_s^L \sin\theta\,ds. \tag{7}$$

$E$ is the Young's modulus, $I$ is the moment of inertia, $k$ is the foundation modulus, $m$ is the mass per unit length, $L$ is the length of a column and $u$ and $v$ are the deflections along $X$- and $Y$-axis, respectively. $\alpha$ is the tip-angle ($\theta = \alpha$ at $s = 0$) and $\gamma\alpha$ is the angle between the force $P$ and the vertical direction with $\gamma$ a constant coefficient. At $s = 0$, Eqs. (6) and (7) give the tip-co-ordinates $(X_a, Y_a)$ of the column.

Inserting the expressions $u = \bar{u}(s)e^{i\Omega t}$ and $v = \bar{v}(s)e^{i\Omega t}$ in Eqs. (2) and (3) and differentiating Eqs. (1)–(7) with respect to $s$, one obtains

$$EI\frac{d^2\theta}{ds^2} + P\sin(\theta - \gamma\alpha) + (m\Omega^2 - k)\left\{\cos\theta\int_0^s Y\,d\xi + \sin\theta\int_0^s (L - \xi - X)\,d\xi\right\} = 0, \quad (8)$$

$$\frac{dX}{ds} + \cos\theta = 0, \quad (9)$$

$$\frac{dY}{ds} + \sin\theta = 0, \quad (10)$$

where $i = \sqrt{-1}$, and $\Omega$ denotes the circular frequency. The boundary conditions for Eqs. (8)–(10) are

$$\frac{d\theta}{ds} = 0 \quad \text{at } s = 0, \quad (11)$$

$$\theta = X = Y = 0 \quad \text{at } s = L. \quad (12)$$

Defining $\eta = s/L$, $x = X/L$ and $y = Y/L$, one can write Eqs. (8)–(12) in non-dimensional form as

$$\theta'' + \lambda\sin(\theta - \gamma\alpha) + (\omega^2 - K)\{V\cos\theta + H\sin\theta\} = 0, \quad (13)$$

$$H' - (1 - \eta - x) = 0, \quad (14)$$

$$V' - y = 0, \quad (15)$$

$$x' + \cos\theta = 0, \quad (16)$$

$$y' + \sin\theta = 0, \quad (17)$$

$$\theta = \alpha,\ \theta' = H = V = 0 \quad \text{at } \eta = 0, \quad (18)$$

$$\theta' = x = y = 0 \quad \text{at } \eta = 1, \quad (19)$$

where $\lambda = PL^2/(EI)$ is the load parameter; $K = kL^4/(EI)$ is the foundation parameter; $\omega = \sqrt{m/(EI)}\,\Omega L^2$, is the frequency parameter; $H = \int_0^\eta (1 - \xi - x)\,d\xi$; $V = \int_0^\eta y\,d\xi$; and a prime denotes differentiation with respect to $\eta$.

In the present analysis, the load versus frequency curve (namely the eigencurve) of the column is essential for studying the stability of the equilibrium position of the column as well as for the large deflections (post-buckling) analysis of the column. In general, static stability loads are those loads at which the eigencurve meets the load axis, whereas the dynamic stability loads are those loads at which the two branches of the eigencurve coalesce. The two-point boundary value problem described by Eqs. (13)–(19) is dependent on the load parameter ($\lambda$), subtangential parameter ($\gamma$), tip-angle ($\alpha$), foundation parameter ($K$) and the frequency parameter ($\omega$). Following the numerical procedure of Ref. [27], the two-point boundary value problem was converted to an initial value problem by estimating the values of $x$ and $y$ at $\eta = 0$ as

$$x = x_a, \ \ y = y_a \tag{20}$$

and the value of $\omega$ for the specified values of $\lambda$, $\alpha$ and $K$ in an iterative process, so as to satisfy the conditions given in Eq. (19). The differential equations were integrated by a fourth order Runge–Kutta integration scheme with a fixed step size of 0.01.

## 3. Results and discussion

The post-critical behaviour of a uniform cantilever column resting on an elastic foundation under a subtangential follower force at its free end is examined here by using the dynamic criterion. In the present analysis, the subtangential parameter $\gamma = 0$, represents the Euler column, whereas, $\gamma = 1$ represents the Becks column. In the Winkler foundation, the elastic medium below the structure is represented by independent linear springs. The surface displacement of the elastic medium at every point is directly proportional to the load applied at that point, and the foundation is modelled by using only one parameter, the stiffness of the Winkler springs. When the column is loaded beyond its critical load, the column deforms both in axial and transverse directions. When the column is resting on an elastic foundation, it is essential to represent the normal reaction by considering terms $ku$ and $kv$ in Eqs. (2) and (3) for $L_{JA}$ and $L_{JT}$. In case of horizontal reaction, the term $ku$ in Eq. (2) is not considered. Some observations of interest obtained from the present analysis are highlighted below.

### 3.1. Large-amplitude free vibrations of cantilever columns

For $\lambda = 0$, the values of $\omega$ obtained from the present numerical computation represent the natural frequencies of the unloaded cantilever column. While studying the large-amplitude vibrations of cantilever beams [26] without elastic foundation ($K = 0$), an increase in the frequency parameter ($\omega$) is observed for the specified amplitude, when the contribution of the axial inertia term ($L_{JA}$) is not taken into account. In the present analysis, both the axial and transverse inertia terms are considered while evaluating the natural frequencies of cantilever columns resting on an elastic foundation. For small values of the tip-angle, $\alpha$ (say, 0.01°), the solution of the problem yields the linear free vibrations of a uniform cantilever column. Table 1 gives variation of natural frequency parameter ($\omega$) with tip-angle ($\alpha$) of unloaded cantilever column resting on an elastic foundation. The frequency parameter increases with tip-angle ($\alpha$) and also with the foundation parameter ($K$). For the specified tip-angle ($\alpha$), the variation of natural frequency with $K$ can be expressed as $\omega = \sqrt{\omega^2(\text{for } K = 0) + K}$.

### 3.2. Post-critical behaviour of Euler columns

In the present analysis, the subtangential parameter $\gamma = 0$, represents the Euler column. For the limiting case of small deflections (i.e., $\alpha \to 0$), $X = L - s$, and the displacement ($u$) along the axis of the column is negligibly small. Hence, the contribution of $L_{JA}$ in Eq. (1) can be neglected as in

Table 1
Large-amplitude vibrations of a uniform cantilever column resting on an elastic foundation

| $\alpha$ (deg) | $x_a$ | $y_a$ | Frequency parameter, $\omega$ | | | |
|---|---|---|---|---|---|---|
| | | | $K = 0$ | $K = 1$ | $K = 10$ | $K = 100$ |
| 0.01 | 1.0000 | 0.0001 | 3.5160 | 3.6555 | 4.7289 | 10.6000 |
| 10 | 0.9907 | 0.1263 | 3.5229 | 3.6621 | 4.7340 | 10.6024 |
| 20 | 0.9629 | 0.2500 | 3.5437 | 3.6821 | 4.7495 | 10.6093 |
| 30 | 0.9172 | 0.3683 | 3.5790 | 3.7161 | 4.7759 | 10.6212 |
| 40 | 0.8545 | 0.4788 | 3.6297 | 3.7649 | 4.8140 | 10.6384 |
| 50 | 0.7761 | 0.5791 | 3.6970 | 3.8299 | 4.8650 | 10.6615 |
| 60 | 0.6834 | 0.6670 | 3.7831 | 3.9130 | 4.9307 | 10.6917 |

Table 2
Post-critical loads of Euler column ($\gamma = 0$) and Beck column ($\gamma = 1$) without foundation ($K = 0$)

| Tip-angle $\alpha$ (deg) | Euler column ($\gamma = 0$) | | | Beck column ($\gamma = 1$) | | | |
|---|---|---|---|---|---|---|---|
| | $\lambda_{cr}$ | $x_a$ | $y_a$ | $\lambda_{cr}$ | $\omega_c$ | $x_a$ | $y_a$ |
| 0.01 | 2.4675 | 1.0000 | 0.0001 | 20.0522 | 11.9202 | 1.0000 | 0.0001 |
| 10 | 2.4768 | 0.9924 | 0.1108 | 20.1889 | 11.0203 | 0.9944 | 0.0748 |
| 20 | 2.5054 | 0.9697 | 0.2194 | 20.6123 | 11.0477 | 0.9776 | 0.1486 |
| 30 | 2.5540 | 0.9324 | 0.3239 | 21.3529 | 11.1104 | 0.9499 | 0.2203 |
| 40 | 2.6245 | 0.8812 | 0.4222 | 22.4690 | 11.2094 | 0.9117 | 0.2891 |
| 50 | 2.7192 | 0.8170 | 0.5126 | 24.0584 | 11.3763 | 0.8635 | 0.3535 |
| 60 | 2.8418 | 0.7410 | 0.5932 | 26.2815 | 11.6209 | 0.8062 | 0.4129 |

Ref. [25] for linear stability analysis. Table 2 gives post-critical loads of Euler column ($\gamma = 0$) without foundation ($K = 0$) for the values of the tip-angle, $\alpha$ varying from 10° to 60°. The post-critical load parameter ($\lambda_{cr}$) increases with tip-angle ($\alpha$) of the column. Table 3 gives the comparison of post-critical load parameter ($\lambda_{cr}$) for Euler column ($\gamma = 0$) resting on an elastic foundation. For the case of horizontal reaction, the load parameter $\lambda_{cr}$ decreases with $\alpha$ when the foundation parameter is higher, and this indicates that the columns incline to the post-critical unstable state (not able to carry more load beyond the critical point) when raising the foundation parameter. In the case of normal reaction, the post-critical load parameter $\lambda_{cr}$ increases with $\alpha$ and also with $K$.

### 3.3. Post-critical behaviour of Beck columns

In the present analysis, the subtangential parameter $\gamma = 1$, represents the Becks column. For the limiting case of small deflections (i.e., $\alpha \to 0$), $X = L - s$, and the displacement ($u$) along the axis of the column is negligibly small. Hence, the contribution of $L_{JA}$ in Eq. (1) can be neglected as in

Table 3
Comparison of post-critical load parameters ($\lambda_{cr}$) for Euler column ($\gamma = 0$) resting on an elastic foundation

| Tip-angle α (deg) | Normal reaction | | | Horizontal reaction | | |
|---|---|---|---|---|---|---|
| | $K = 1$ | $K = 10$ | $K = 100$ | $K = 1$ | $K = 10$ | $K = 100$ |
| 0.01 | 2.6500 | 4.1783 | 11.9966 | 2.6500 | 4.1783 | 11.9966 |
| 10 | 2.6594 | 4.1886 | 12.0277 | 2.6580 | 4.1779 | 11.9877 |
| 20 | 2.6880 | 4.2197 | 12.1219 | 2.6827 | 4.1772 | 11.9613 |
| 30 | 2.7368 | 4.2726 | 12.2806 | 2.7249 | 4.1766 | 11.9165 |
| 40 | 2.8075 | 4.3490 | 12.5061 | 2.7862 | 4.1773 | 11.8522 |
| 50 | 2.9025 | 4.4511 | 12.8023 | 2.8694 | 4.1810 | 11.7665 |
| 60 | 3.0253 | 4.5825 | 13.1746 | 2.9781 | 4.1904 | 11.6568 |

Table 4
Comparison of post-critical load parameters ($\lambda_{cr}$) for the Beck column ($\gamma = 1$) resting on an elastic foundation

| Tip-angle | Normal reaction | Horizontal reaction | | |
|---|---|---|---|---|
| α (deg) | $K = 1, 10, 100$ | $K = 1$ | $K = 10$ | $K = 100$ |
| 0.01 | 20.0522 | 20.0522 | 20.0521 | 20.0519 |
| 10 | 20.1888 | 20.1876 | 20.1765 | 20.0667 |
| 20 | 20.6123 | 20.6073 | 20.5627 | 20.1212 |
| 30 | 21.3529 | 21.3414 | 21.2385 | 20.2371 |
| 40 | 22.4690 | 22.4479 | 22.2588 | 20.4532 |
| 50 | 24.0584 | 24.0238 | 23.7143 | 20.8280 |
| 60 | 26.2815 | 26.2282 | 25.7538 | 21.4480 |

Ref. [25] for linear stability analysis. The post-critical loads of Beck column ($\gamma = 1$) without foundation ($K = 0$) for the values of the tip-angle α varying from 10° to 60° are also presented in Table 2. The critical load parameter ($\lambda_{cr}$) increases with tip-angle (α) of the column. Table 4 gives the comparison of post-critical load parameters ($\lambda_{cr}$) for Beck column ($\gamma = 1$) resting on an elastic foundation. It is very interesting to note that the post-critical load parameter $\lambda_{cr}$ increases with α for both the cases of horizontal reaction and normal reaction. For the case of horizontal reaction, $\lambda_{cr}$ decreases with increase in the foundation parameter ($K$), whereas there is no change in $\lambda_{cr}$ values when raising the foundation parameter for the case of normal reaction. Hence, a uniform cantilever column subjected to a follower force ($\gamma = 1$) and having dynamic instability is neither stabilized nor destabilized by the introduction of a uniform elastic foundation for the case of normal reaction.

## 4. Conclusions

Post-critical behaviour of Euler and Beck columns resting on an elastic foundation has been examined by considering the realistic normal foundation reaction. The formulation of the problem is general (which is valid for all subtangential follower force) and it can be extended easily for non-uniform cantilever columns.

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